

On classical solutions to elliptic boundary value problems. The full regularity spaces $C_{\alpha}^{0,\lambda}(\overline{\Omega})$.

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Abstract

Let L be a second order, uniformly elliptic operator, and consider the equation $-Lu = f$ under the homogeneous boundary condition $u = 0$. It is well known that $f \in C(\overline{\Omega})$ does not guarantee $\nabla^2 u \in C(\overline{\Omega})$. This gap led to look for functional spaces $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$, as large as possible, for which $f \in C_*(\overline{\Omega})$ *merely* guarantees the continuity of $\nabla^2 u$ (but nothing more, say). Hölder continuity is too restrictive to fulfill this minimal requirement since in this case $\nabla^2 u$ inherits the whole regularity enjoyed by f (we say that *full regularity* occurs). This two opposite situations led us to look for significant cases in which *intermediate regularity* (i.e., between *mere continuity* and *full regularity*) occurs. This holds for data in Log spaces $D^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < +\infty$, simply obtained by replacing in the modulus of continuity of Hölder spaces the quantity $1/|x - y|$ by $\log(1/|x - y|)$. If $f \in D^{0,\alpha}$, for some fixed $\alpha > 1$, then $\nabla^2 u \in D^{0,\alpha-1}$. This regularity is optimal.

The above picture opened the way to further investigation. Below we study the more general problem of data f in subspaces of continuous functions $D_{\overline{\omega}}$, characterized by a given *modulus of continuity* $\overline{\omega}(r)$. Hölder and Log spaces are particular cases. A significant new, lets say curious, case is shown by the family of functional spaces $C_{\alpha}^{0,\lambda}(\overline{\Omega})$, $0 \leq \lambda < 1$, $\alpha \in \mathbb{R}$. In particular, $C_0^{0,\lambda}(\overline{\Omega}) = C^{0,\lambda}(\overline{\Omega})$, and $C_{\alpha}^{0,0}(\overline{\Omega}) = D^{0,\alpha}(\overline{\Omega})$. Main point is that full regularity occurs for $\lambda > 0$ and arbitrary $\alpha \in \mathbb{R}$. If $f \in C_{\alpha}^{0,\lambda}(\overline{\Omega})$ then $\nabla^2 u \in C_{\alpha}^{0,\lambda}(\overline{\Omega})$.

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1 Introduction.

We start by some notation. By Ω we denote an open, bounded, connected set in \mathbb{R}^n , locally situated on one side of its boundary Γ . The boundary Γ is of class $C^{2,\lambda}$, for some λ , $0 < \lambda \leq 1$. Notation $\Omega_0 \subset\subset \Omega$ means that the open set Ω_0 satisfies the property $\overline{\Omega}_0 \subset \Omega$.

By $C(\overline{\Omega})$ we denote the Banach space of all real continuous functions f defined in $\overline{\Omega}$. The "sup" norm is denoted by $\|f\|$. We also appeal to the classical spaces $C^k(\overline{\Omega})$ endowed with their usual norms $\|u\|_k$, and to the Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, endowed with the standard semi-norms and norms. The space

$C^{0,1}(\overline{\Omega})$, is sometimes denoted by $Lip(\overline{\Omega})$, the space of Lipschitz continuous functions in $\overline{\Omega}$. We set

$$I(x; r) = \{y : |y - x| \leq r\}, \quad \Omega(x; r) = \Omega \cap I(x; r).$$

Symbols c and C denote generical positive constants. We may use the same symbol to denote different constants.

Let us present some reasons that led us to the present study. We say that solutions to a specific boundary value problem are *classical* if all derivatives appearing in the equations and boundary conditions are continuous up to the boundary on their domain of definition. We call "*minimal assumptions problem*" the investigation of "minimal assumptions" on the data which guarantee that solutions are classical. The very starting point of these notes was reference [1], where the main goal was to look for *minimal assumptions* on the data which guarantee classical solutions to the 2-D Euler equations in a bounded domain. The study of this problem led to the auxiliary problem

$$(1.1) \quad \begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

We do not discuss here the relation between the Euler equations and problem (1.1). The interested reader is referred to the original paper [1], and also to [4], where a complete description is presented.

Below we consider second order, uniformly elliptic operators

$$(1.2) \quad L = \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j.$$

Without loss of generality, we assume that the matrix of coefficients is symmetric. To avoid conditions depending on the single case, we assume once and for all that the operator's coefficients are Lipschitz continuous in $\overline{\Omega}$. Lower order terms can be considered without difficulty.

A Hölder continuity assumption on f is unnecessarily restrictive to guarantee $\nabla^2 u \in C(\overline{\Omega})$, where u is the solution to problem (1.1). On the other hand, continuity of f is not sufficient to guarantee continuity of $\nabla^2 u$. This situation led us to consider in [1] a Banach space $C_*(\overline{\Omega})$, $C^{0,\lambda}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset C(\overline{\Omega})$, for which the following result holds (Theorem 4.5, in [1]).

Theorem 1.1. *Let $f \in C_*(\overline{\Omega})$ and let u be the solution of problem (1.1). Then $u \in C^2(\overline{\Omega})$, moreover,*

$$(1.3) \quad \|\nabla^2 u\| \leq c \|f\|_*.$$

The above result was stated for constant coefficients operators, however the proof applies without any modification to variable coefficients case, since it is based on some properties of the Green functions, which hold in the general case.

For the readers convenience we recall definition and main properties of $C_*(\overline{\Omega})$ (see [1] and, for complete proofs, [2]). Define, for $f \in C(\overline{\Omega})$, and for each $r > 0$,

$$(1.4) \quad \omega_f(r) \equiv \sup_{x, y \in \Omega; 0 < |x - y| \leq r} |f(x) - f(y)|,$$

and consider the semi-norm

$$(1.5) \quad [f]_* = [f]_{*,R} \equiv \int_0^R \omega_f(r) \frac{dr}{r},$$

where $R > 0$ is fixed. The finiteness of the above integral is known as *Dini's continuity condition*. We define the functional space

$$(1.6) \quad C_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_* < \infty\}$$

normalized by $\|f\|_* = [f]_* + \|f\|$. Norms defined for two distinct values of R are equivalent. We have shown that $C_*(\overline{\Omega})$ is a Banach space, that the embedding $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact, and that the set $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$.

The regularity theorem 1.1 for data in $C_*(\overline{\Omega})$ raise a number of new questions. Contrary to the case of Hölder continuity, where full regularity is restored ($\nabla^2 u$ and f has the same regularity), no significant additional regularity is obtained for data in $C_*(\overline{\Omega})$, besides mere continuity of $\nabla^2 u$. So, we are in the presence of two totally opposite behaviors. An "intermediate" situation is shown by the Log spaces $D^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < +\infty$. In the constant coefficients case, if $f \in D_{loc}^{0,\alpha}(\Omega)$ for fixed $\alpha > 1$, then $\nabla^2 u \in D_{loc}^{0,\alpha-1}(\Omega)$. This regularity result is *optimal*. Furthermore, it holds up to "flat boundary points". See theorem 9.1 below.

The above picture leads us to consider general data spaces $D_{\overline{\omega}}(\overline{\Omega})$, characterized by a given *modulus of continuity* function $\overline{\omega}(r)$. These spaces are contained between $Lip(\overline{\Omega})$ and $C_*(\overline{\Omega})$. Hölder and Log spaces are particular cases. To each suitable $\overline{\omega}(r)$ there corresponds a $\widehat{\omega}(r)$ such that $\nabla^2 u \in D_{\widehat{\omega}}$ for $f \in D_{\overline{\omega}}$, see theorem 3.2. Clearly, $\overline{\omega}(r) \leq c\widehat{\omega}(r)$, for some $c > 0$. This situation occurs for data in Log spaces, see theorem 8.1. Furthermore, if a reverse inequality $\widehat{\omega}(r) \leq c\overline{\omega}(r)$ holds, then full regularity occurs, see theorem 3.3. This is the situation for data in Hölder spaces. A more general, quite significant, case of full regularity concerns the new family of functional spaces $C_\alpha^{0,\lambda}(\overline{\Omega})$, $0 \leq \lambda < 1$, $\alpha \in \mathbb{R}$, called here Hölog spaces. For $\lambda > 0$ and $\alpha = 0$, $C_0^{0,\lambda}(\overline{\Omega}) = C^{0,\lambda}(\overline{\Omega})$, is a Hölder classical space. For $\lambda = 0$ and $\alpha > 0$, $C_\alpha^{0,0}(\overline{\Omega}) = D^{0,\alpha}(\overline{\Omega})$ is a Log space. Main point is that, for $\lambda > 0$, $\nabla^2 u$ and f enjoy the same $C_\alpha^{0,\lambda}(\overline{\Omega})$ regularity (full regularity). See theorem 9.1.

The assumptions on the data spaces $D_{\overline{\omega}}(\overline{\Omega})$ required in theorems 3.2 and 3.3 can be substantially weakened. However, explicit statements in this direction would not add particularly significant features, at the cost of more involved manipulations.

2 The spaces $D_{\overline{\omega}}(\overline{\Omega})$. General properties.

In this section we define the spaces $D_{\overline{\omega}}(\overline{\Omega})$ and state some general properties. We consider real, *continuous, non-decreasing* functions $\overline{\omega}(r)$, defined for $0 \leq r < R$, for some $R > 0$. Furthermore, $\overline{\omega}(0) = 0$, and $\overline{\omega}(0) > 0$ for $r > 0$. These three conditions are assumed everywhere in the sequel. Sometimes, the functions $\overline{\omega}(r)$ will be called *oscillation functions*.

Recalling (1.4), we set

$$(2.1) \quad [f]_{\overline{\omega}} = \sup_{0 < r < R} \frac{\omega_f(r)}{\overline{\omega}(r)}.$$

Hence,

$$(2.2) \quad \omega_f(r) \leq [f]_{\overline{\omega}} \overline{\omega}(r), \quad \forall r \in (0, R).$$

Further, we define the linear space

$$(2.3) \quad D_{\overline{\omega}}(\overline{\Omega}) = \{ f \in C(\overline{\Omega}) : [f]_{\overline{\omega}} < \infty \}.$$

One easily shows that $[f]_{\overline{\omega}}$ is a semi-norm in $D_{\overline{\omega}}(\overline{\Omega})$. We introduce a norm by setting

$$(2.4) \quad \|f\|_{\overline{\omega}} = [f]_{\overline{\omega}} + \|f\|.$$

Two norms with distinct values of the parameter R are equivalent, due to the addition of $\|f\|$ to the semi-norms.

It is worth noting that, beyond the three conditions on $\overline{\omega}(r)$ introduced above, any other property assumed in the sequel is merely needed in an arbitrarily small neighborhood of the origin. This fact may be used without a continual reference. In the sequel, to avoid continual specification, we introduce the following definitions.

Definition 2.1. *We say that $\overline{\omega}(r)$ is concave if it is concave in a neighborhood of the origin, and say that $\overline{\omega}(r)$ is differentiable if it is point-wisely differentiable (not necessarily continuously differentiable), for each $r > 0$, in a neighborhood of the origin.*

Next we establish some useful properties of the above functional spaces.

Proposition 2.1. *If*

$$(2.5) \quad 0 < k_0 \leq \frac{\overline{\omega}(r)}{\overline{\omega}_0(r)} \leq k_1 < +\infty,$$

for r in some neighborhood of the origin, then $D_{\overline{\omega}}(\overline{\Omega}) = D_{\overline{\omega}_0}(\overline{\Omega})$, with equivalent norms.

The proof is immediate.

Lemma 2.1. *If $\|f_n\|_{\overline{\omega}} \leq C_0$, and $f_n \rightarrow f$ in $C(\overline{\Omega})$ then $\|f\|_{\overline{\omega}} \leq C_0$.*

The proof is immediate.

Theorem 2.2. *$D_{\overline{\omega}}(\overline{\Omega})$ is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $D_{\overline{\omega}}(\overline{\Omega})$. It follows, in particular, that $f_n \rightarrow f$ in $C(\overline{\Omega})$, where $f \in D_{\overline{\omega}}(\overline{\Omega})$. On the other hand, for $|x - y| = r$,

$$\begin{aligned} & \frac{|(f(x) - f_n(x)) - (f(y) - f_n(y))|}{\overline{\omega}(r)} = \\ & \lim_{m \rightarrow \infty} \frac{|(f_m(x) - f_n(x)) - (f_m(y) - f_n(y))|}{\overline{\omega}(r)} \leq \limsup_{m \rightarrow \infty} [f_m - f_n]_{\overline{\omega}}. \end{aligned}$$

Hence

$$[f - f_n]_{\bar{\omega}} \leq \limsup_{m \rightarrow \infty} [f_m - f_n]_{\bar{\omega}}.$$

From the Cauchy sequence hypothesis it readily follows that

$$\lim_{n \rightarrow \infty} [f - f_n]_{\bar{\omega}} = 0.$$

□

Next we consider compact embedding properties. In the sequel, $\bar{\omega} \ll \bar{\omega}_1$ mean that

$$(2.6) \quad \lim_{r \rightarrow 0} \frac{\bar{\omega}(r)}{\bar{\omega}_1(r)} = 0.$$

Theorem 2.3. *Assume that $\bar{\omega} \ll \bar{\omega}_1$. Then the embedding*

$$D_{\bar{\omega}}(\bar{\Omega}) \subset D_{\bar{\omega}_1}(\bar{\Omega}),$$

is compact.

Proof. By assumption

$$\|f_n\|_{\bar{\omega}} = [f_n]_{\bar{\omega}} + \|f_n\| \leq C_0, \quad \forall n.$$

From (2.6) it follows that $\bar{\omega}(r) \leq \bar{\omega}_1(r)$ for $r \in (0, R_0)$, for some $R_0 > 0$. For $r \in (R_0, R)$ one has $\bar{\omega}(r) \leq \frac{\bar{\omega}(R)}{\bar{\omega}_1(R_0)} \bar{\omega}_1(r)$. So there is a positive constant C such that

$$\bar{\omega}(r) \leq C \bar{\omega}_1(r), \quad \forall r \in (0, R).$$

By the Ascoli-Arzelà Theorem, the embedding

$$D_{\bar{\omega}}(\bar{\Omega}) \subset C(\bar{\Omega})$$

is compact. Hence, by appealing to lemma 2.1, one shows that there is a subsequence, still denoted f_n , which converges uniformly to some $f \in D_{\bar{\omega}}(\bar{\Omega})$. Without loss of generality, we assume that $f = 0$.

Let $|x - y| = r$. One has

$$\frac{|f_n(x) - f_n(y)|}{\bar{\omega}_1(r)} = \frac{|f_n(x) - f_n(y)|}{\bar{\omega}(r)} \frac{\bar{\omega}(r)}{\bar{\omega}_1(r)}, \quad \forall n.$$

Given $\epsilon > 0$, it follows from (2.6) that there is $R_0(\epsilon) > 0$ such that

$$(2.7) \quad 0 < r \leq R_0(\epsilon) \implies \frac{\bar{\omega}(r)}{\bar{\omega}_1(r)} < \epsilon.$$

Hence, for $0 < |x - y| \leq R_0(\epsilon)$,

$$(2.8) \quad \frac{|f_n(x) - f_n(y)|}{\bar{\omega}_1(r)} \leq C_0 \epsilon, \quad \forall n.$$

On the other hand, if $r \in (R_0(\epsilon), R)$, one has

$$\frac{|f_n(x) - f_n(y)|}{\bar{\omega}_1(r)} \leq \frac{2}{\bar{\omega}_1(R_0(\epsilon))} \|f_n\|.$$

Since the sequence $\|f_n\|$ converges to zero, there is an index $N(\epsilon)$ such that, for each $n > N(\epsilon)$, the right hand side of the last inequality is smaller than ϵ . This fact, together with (2.8), shows that (2.8) holds for $0 < |x - y| \leq R$ and $n > N(\epsilon)$ (increase the constant C_0 , if necessary). So,

$$\lim_{n \rightarrow +\infty} [f_n]_{\overline{\omega}} = 0.$$

□

Lemma 2.4. *Assume that $\overline{\omega}$ is concave. Then*

$$(2.9) \quad \overline{\omega}(kr) \leq k \overline{\omega}(r), \quad \forall k \geq 1.$$

The proof is immediate.

In reference [5], Theorem 4.4, we claimed that $C^\infty(\overline{\Omega})$ is dense in Log spaces, leaving the proof to the reader. This result is wrong, as shown below. It is worth noting that $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$, a result that has a central role in reference [2].

Theorem 2.5. *Assume that $\overline{\omega}(r)$ is concave and that $\overline{\omega}_1(r) << \overline{\omega}(r)$. Then $D_{\overline{\omega}_1}(\overline{\Omega})$ is not dense in $D_{\overline{\omega}}(\overline{\Omega})$.*

Proof. We assume that the origin belongs to Ω , and argue in a neighborhood $I = I(0, \delta) \subset \Omega$. Define f by setting $f(x) = \overline{\omega}(|x|)$. We show that $[f - g]_{\overline{\omega}} \geq 1$, for each $g \in D_{\overline{\omega}_1}(\overline{\Omega})$. It is sufficient to consider the one-dimensional case. One has

$$\frac{|(f(x) - g(x)) - (f(0) - g(0))|}{\overline{\omega}(|x|)} = \left| 1 - \frac{g(x) - g(0)}{\overline{\omega}(|x|)} \right|.$$

Hence $[f - g]_{\overline{\omega}} \geq 1$ follows, if we show that

$$(2.10) \quad \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{\overline{\omega}(|x|)} = 0.$$

Let's prove this last inequality. One has, as $x \rightarrow 0$,

$$(2.11) \quad \lim \frac{g(x) - g(0)}{\overline{\omega}(|x|)} = \lim \frac{g(x) - g(0)}{\overline{\omega}_1(|x|)} \cdot \lim \frac{\overline{\omega}_1(|x|)}{\overline{\omega}(|x|)} = 0.$$

□

Note that in the above proof we did not explicitly appeal to the concavity assumption. This assumption was introduced merely to guarantee that $f(x) = \overline{\omega}(|x|)$ belongs to $D_{\overline{\omega}}$ in a neighborhood of the origin. This holds if

$$(2.12) \quad \overline{\omega}(s) \leq \overline{\omega}(r) + c \overline{\omega}(s - r), \quad \text{for } 0 < r < s < \rho,$$

for some constant $c \geq 1$, and some $\rho > 0$. By lemma 2.4, concave oscillation functions satisfy (2.12) with $c = 1$.

The above result shows, in particular, that $C^{0,\mu}(\overline{\Omega})$ is not dense in $C^{0,\lambda}(\overline{\Omega})$ for $1 \geq \mu > \lambda > 0$. In particular $Lip(\overline{\Omega})$, hence $C^1(\overline{\Omega})$, is not dense in $C^{0,\lambda}(\overline{\Omega})$.

We end this section by stating an extension theorem, where $\Omega_\delta \equiv \{x : \text{dist}(x, \Omega) < \delta\}$.

Theorem 2.6. *Assume that Ω is convex or, alternatively, that $\bar{\omega}(r)$ is concave (concavity may be replaced by condition (3.16)). Then there is a $\delta > 0$ such that the following holds. There is a linear continuous map T from $C(\bar{\Omega})$ to $C(\bar{\Omega}_\delta)$, and from $D_{\bar{\omega}}(\bar{\Omega})$ to $D_{\bar{\omega}}(\bar{\Omega}_\delta)$, such that $Tf(x) = f(x)$, for each $x \in \bar{\Omega}$.*

The proof follows by appealing to the argument used to prove the Theorem 2.3 in [2]. See reference [5]. Note that the classical proof of approximation of functions on compact subsets of Ω by appealing to mollification, does not work here. Otherwise, the density property refused by theorem 2.5 would hold.

3 Spaces $D_{\hat{\omega}}(\bar{\Omega})$ and regularity. The main theorems.

In this section we state the theorems 3.2 and 3.3. From now on we assume that the modulus of continuity $\bar{\omega}(r)$ satisfy the condition

$$(3.1) \quad \int_0^R \bar{\omega}(r) \frac{dr}{r} \leq C_R,$$

for some constant C_R . Assumption (3.1) is equivalent to the inclusion $D_{\bar{\omega}}(\bar{\Omega}) \subset C_*(\bar{\Omega})$. This assumption is almost necessary to obtain $\nabla^2 u \in C(\bar{\Omega})$.

We put each suitable oscillation function $\bar{\omega}(r)$ in correspondence with a unique, related oscillation function $\hat{\omega}(r)$ defined by setting $\hat{\omega}(0) = 0$, and

$$(3.2) \quad \hat{\omega}(r) = \int_0^r \bar{\omega}(s) \frac{ds}{s}$$

for $0 < r \leq R$. Hence, to a functional space $D_{\bar{\omega}}(\bar{\Omega})$ there corresponds a well defined functional space $D_{\hat{\omega}}(\bar{\Omega})$. Obviously, $\hat{\omega}$ satisfies all the properties described in section 2 for generical oscillation functions. In particular, Banach spaces

$$(3.3) \quad D_{\hat{\omega}}(\bar{\Omega}) = \{ f \in C(\bar{\Omega}) : [f]_{\hat{\omega}} < \infty \}$$

turn out to be well defined.

Next we discuss some additional restrictions on the data spaces $D_{\bar{\omega}}(\bar{\Omega})$. We start by excluding $Lip(\bar{\Omega})$ as data space since this *singular* case, largely considered in literature, is borderline. In fact, to assign $f \in Lip(\bar{\Omega})$ is equivalent to assign $\nabla f \in L^\infty(\Omega)$, which is the starting point of a new chapter. So, we impose the *strict* limitation

$$(3.4) \quad Lip(\bar{\Omega}) \subset D_{\bar{\omega}}(\bar{\Omega}) \subset C_*(\bar{\Omega}).$$

Exclusion of $Lip(\bar{\Omega})$ means that $\bar{\omega}(r)$ does not verify $\bar{\omega}(r) \leq cr$, for any positive constant c . Hence $\limsup(\bar{\omega}(r)/r) = +\infty$, as $r \rightarrow 0$. We simplify, by assuming that

$$(3.5) \quad \lim_{r \rightarrow 0} \frac{\bar{\omega}(r)}{r} = +\infty.$$

In particular, the graph of $\bar{\omega}(r)$ is tangent to the vertical axis at the origin (as for Hölder and Log spaces). This picture also shows that *concavity* of the

graph is here a quite natural assumption. Concavity implies that left and right derivatives are well defined, for $r > 0$. By also taking into account that $\bar{\omega}(r)$ is non-decreasing, we realize that pointwise differentiability of $\bar{\omega}(r)$, for $r > 0$, is not a particularly restrictive assumption. This last claim is reenforced by the equivalence result for norms, under condition (2.5). This equivalence allows regularization of oscillation functions $\bar{\omega}(r)$, staying inside the same original functional space $D_{\bar{\omega}}(\bar{\Omega})$. Summarizing, *differentiability* and *concavity* (recall definition 2.1) are natural assumptions here.

If $\bar{\omega}(r)$ is concave, not flat, and differentiable, it follows that

$$(3.6) \quad \frac{\bar{\omega}(r)}{r \bar{\omega}'(r)} > 1,$$

for all $r > 0$. This justifies the assumption

$$(3.7) \quad \lim_{r \rightarrow 0} \frac{\bar{\omega}(r)}{r \bar{\omega}'(r)} = C_1 > 1,$$

where $C_1 = +\infty$ is admissible. Assumption (3.7) is reenforced by the particular situation in Lipschitz, Hölder, and Log cases. The limit exists and is given by, respectively, 1 , $\frac{1}{\lambda}$, and $+\infty$. As expected, the Lipschitz case stays outside the admissible range. Note that, basically, the larger is the space, the larger is the limit.

The above consideration allow us to assume in theorems 3.2 and 3.3 that oscillation functions $\bar{\omega}(r)$, are concave, differentiable, and satisfies conditions (3.1), (3.5), and (3.7).

Note that, due to a possible loss of regularity, it could happen that a $D_{\hat{\omega}}(\bar{\Omega})$ space is not contained in $C_*(\bar{\Omega})$, as happens in theorem 8.1, if $1 < \alpha < 2$. In other words, $\hat{\omega}(r)$ does not necessarily satisfy (3.1).

Next, we define the quantity

$$(3.8) \quad B(r) =: \frac{r \int_r^R \frac{\bar{\omega}(s)}{s^2} ds}{\int_0^r \frac{\bar{\omega}(s)}{s} ds}.$$

The following result holds.

Lemma 3.1. *Assume that $\bar{\omega}(r)$ is concave and satisfies assumptions (3.1), (3.5) and (3.7). Then*

$$(3.9) \quad \lim_{r \rightarrow 0} B(r) = \frac{1}{C_1 - 1}.$$

In particular there is a positive constant C_2 such that

$$(3.10) \quad B(r) \leq C_2$$

in some neighborhood of the origin.

Proof. By appealing to (3.1), (3.5) and to a de L'Hôpital's rule one shows that

$$(3.11) \quad \lim_{r \rightarrow 0} \frac{1}{r} \int_0^r \frac{\bar{\omega}(s)}{s} ds = +\infty.$$

On the other hand

$$(3.12) \quad \lim_{r \rightarrow 0} B(r) = \lim_{r \rightarrow 0} \frac{\int_r^R \frac{\overline{\omega}(s)}{s^2} ds}{\frac{1}{r} \int_0^r \frac{\overline{\omega}(s)}{s} ds}.$$

Equation (3.11) shows that the denominator $g(r)$ of the fraction in the right hand side of (3.12) goes to $+\infty$ as r goes to zero. Furthermore its derivative

$$g'(r) = \frac{1}{r^2} \left(\overline{\omega}(r) - \int_0^r \frac{\overline{\omega}(s)}{s} ds \right)$$

is strictly negative for positive r in a neighborhood of the origin, as follows from the inequality $\overline{\omega}(r) - \int_0^r \frac{\overline{\omega}(s)}{s} ds < 0$, for $r > 0$. Let's show this last inequality. Since the left hand side of the inequality goes to zero with r , it is sufficient to show that its derivative is strictly negative for $r > 0$. This follows easily by appealing to (3.7). The above results allow us to apply to the limit (3.12) one of the well known forms of de L'Hôpital's rule. Straightforward calculations, together with (3.7), show (3.9). \square

Next we state our main results, theorems 3.2 and 3.3. In the first theorem constant coefficients are assumed.

Theorem 3.2. *Assume that the oscillation function $\overline{\omega}(r)$, concave and differentiable, satisfies conditions (3.1), (3.5), and (3.7). Define $\widehat{\omega}(r)$ by (3.2). Let $\Omega_0 \subset \subset \Omega$, $f \in D_{\overline{\omega}}(\overline{\Omega})$, and u be the solution of problem (1.1), where the operator coefficients are constant. Then $\nabla^2 u \in D_{\widehat{\omega}}(\Omega_0)$ and*

$$(3.13) \quad \|\nabla^2 u\|_{\widehat{\omega}, \Omega_0} \leq C \|f\|_{\overline{\omega}},$$

for some positive constant $C = C(\Omega_0, \Omega)$. The result is optimal in the sharp sense defined in section 10. Furthermore, the above regularity holds up to flat boundary points.

A point $x \in \partial\Omega$ is said to be a *flat boundary point* if the boundary is flat in a neighborhood of the point. The meaning of *sharp optimality* is the following (our abbreviate notation seems clear).

Definition 3.1. *We say that a given regularity statement of type $\overline{\omega} \rightarrow \widehat{\omega}$ is sharp if any regularity statement $\overline{\omega} \rightarrow \widehat{\omega}_0$, obtained by replacing $\widehat{\omega}$ by any other $\widehat{\omega}_0$, implies the existence of a constant c for which $\widehat{\omega}(r) \leq c\widehat{\omega}_0(r)$.*

The sharp regularity claimed in theorem 3.2 will be proved in section 10.

Much stronger results hold if the constant C_1 in equation (3.7) is positive and finite. In this case one has

$$(3.14) \quad D_{\widehat{\omega}}(\overline{\Omega}) = D_{\overline{\omega}}(\overline{\Omega}).$$

In fact, by de l'Hôpital rule, one shows that

$$\lim_{r \rightarrow 0} \frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r \rightarrow 0} \frac{\overline{\omega}(r)}{r \overline{\omega}'(r)},$$

if the second limit exists. Hence, under this last hypothesis, the identity (3.14) holds if (actually, and only if) the limit is positive and finite. Clearly, (3.14)

holds by merely assuming the inequality required in proposition 2.1. We will show that if (3.14) holds then the operator \mathbf{L} can have variable coefficients, and full regularity occurs up to any (regular) boundary point. More precisely, one has the following result.

Theorem 3.3. *Assume that the oscillation function $\bar{\omega}(r)$, concave and differentiable, satisfies conditions (3.1), (3.5), and (3.7) for some $C_1 < +\infty$. Further, define $\hat{\omega}(r)$ by (3.2). Let $f \in D_{\bar{\omega}}(\bar{\Omega})$, and let u be the solution of problem (1.1). Then $\nabla^2 u \in D_{\bar{\omega}}(\bar{\Omega})$ and*

$$(3.15) \quad \|\nabla^2 u\|_{\bar{\omega}} \leq C \|f\|_{\bar{\omega}},$$

for some positive constant C . Regularity in the sharp sense holds.

Regularity in the sharp sense follows trivially from full regularity. But it is quite significant, even necessary, in dealing with intermediate regularity results, like in theorem 3.2. See the example shown in section 8, in the framework of Log spaces $D^{0,\alpha}(\bar{\Omega})$.

The conditions imposed in the above statements can be weakened as follows. We start by replacing the concavity assumption by the existence of a constant $k_1 > 1$ such that

$$(3.16) \quad \bar{\omega}(k_1 r) \leq c_1 \bar{\omega}(r)$$

for some positive constant c_1 , and for r in a neighborhood of the origin. We take into account that, if (3.16) holds, then given $k_2 > 1$, there is a positive constant c_2 such that

$$(3.17) \quad \bar{\omega}(k_2 r) \leq c_2 \bar{\omega}(r),$$

for r in some δ_0 -neighborhood of the origin. The proof is obvious, by a bootstrap argument. Take into account that, if $k_2 > k_1$, there is an integer m such that $k_2 \leq k_1^m$. If $\bar{\omega}(r)$ is concave the lemma 2.4 shows (3.16) for $k_2 = c_2 = 1$.

Actually, in the sequel we will prove that in theorem 3.2, concavity, differentiability, and assumptions (3.1), (3.5), and (3.7), can be replaced by the more general set of assumptions (3.1), (3.5), (3.16), and (3.10). The same holds for theorem 3.15, by adding the assumption (6.2).

For previous related results we refer to [8] and [13]. The author is grateful to Piero Marcati who, after a seminar on our results, found the above references.

4 An H-K-L-G inequality.

In this section we prove the Theorem 4.1 below. The proof is an adaptation of that developed in [7] to prove the so called Hölder-Korn-Lichtenstein-Giraud inequality (see [7], part II, section 5, appendix 1) in the framework of Hölder spaces. Following [7], we considered *singular kernels* $\mathcal{K}(x)$ of the form

$$(4.1) \quad \mathcal{K}(x) = \frac{\sigma(x)}{|x|^n},$$

where $\sigma(x)$ is infinitely differentiable for $x \neq 0$, and satisfies the properties $\sigma(tx) = \sigma(x)$, for $t > 0$, and

$$\int_S \sigma(x) dS = 0,$$

where $S = \{x : |x| = 1\}$. It follows easily that, for $0 < \rho_1 < \rho_2$,

$$(4.2) \quad \int_{\rho_1 < |x| < \rho_2} \mathcal{K}(x) dx = \int_{\rho_1 < |x|} \mathcal{K}(x) dx = \int \mathcal{K}(x) dx = 0,$$

where the last integral is in the Cauchy principal value sense.

For continuous functions ϕ with compact support, the convolution integral

$$(4.3) \quad (\mathcal{K} * \phi)(x) = \int \mathcal{K}(x - y) \phi(y) dy,$$

extended to the whole space \mathbb{R}^n , exists as a Cauchy principal value and is finite.

We set $I(\rho) = \{x : |x| \leq \rho\}$, $D_{\bar{\omega}}(\rho) = D_{\bar{\omega}}(I(\rho))$, and do the same for other functional spaces, norms, and semi-norms labeled by ρ .

Theorem 4.1. *Let $\mathcal{K}(x)$ be a singular kernel enjoying the properties described above. Further, assume that the oscillation function $\bar{\omega}$ satisfies (3.1), (3.5), (3.16), and (3.10). Let $\phi \in D_{\bar{\omega}}(\rho)$, vanish for $|x| \geq \rho$. Then $\mathcal{K} * \phi \in D_{\bar{\omega}}(\rho)$. Furthermore, in the sphere $I(\rho)$, one has*

$$(4.4) \quad [(\mathcal{K} * \phi)]_{\bar{\omega}} \leq C \|\phi\|_{\bar{\omega}},$$

where $C = C(n, \bar{\omega}, \|\sigma\|)$.

Proof. Let $x_0, x_1 \in I(\rho)$, $0 < |x_0 - x_1| = \delta < \delta_0 \leq \rho$. The positive constant δ_0 is fixed here in correspondence to the choice $k_2 = 3$ in (3.17). In the concave case (assumed, for clearness, in the statements of theorems 3.2 and 3.3), we may set $k_2 = 1$.

For convenience, we will use the simplified notation $\bar{\omega}(r) = \bar{\omega}_{\phi}(r)$. From (4.2) it follows that

$$(\mathcal{K} * \phi)(x) = \int (\phi(y) - \phi(x)) \mathcal{K}(x - y) dy.$$

Hence, with abbreviated notation,

$$(4.5) \quad \begin{aligned} & (\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1) = \\ & \int \left\{ (\phi(y) - \phi(x_0)) \mathcal{K}(x_0 - y) - (\phi(y) - \phi(x_1)) \mathcal{K}(x_1 - y) \right\} dy = \\ & \int_{|y-x_0| < 2\delta} \{\dots\} dy + \int_{2\delta < |y-x_0| < \delta_0} \{\dots\} dy + \int_{\delta_0 < |y-x_0|} \{\dots\} dy \equiv I_1 + I_2 + I_3. \end{aligned}$$

Since

$$\{y : |y - x_1| < 2\delta\} \subset \{y : |y - x_0| < 3\delta\}$$

it follows that

$$\begin{aligned}
& \int_{|y-x_0|<2\delta} |\phi(y) - \phi(x_1)| |\mathcal{K}(x_1 - y)| dy \leq \\
(4.6) \quad & \int_{|y-x_1|<3\delta} |\phi(y) - \phi(x_1)| |\mathcal{K}(x_1 - y)| dy \leq \\
& \|\sigma\| \int_0^{3\delta} \frac{\overline{\omega}(r)}{r} dr \leq \|\sigma\| [\phi]_{\overline{\omega}} \int_0^{3\delta} \frac{\overline{\omega}(r)}{r} dr,
\end{aligned}$$

where we appealed to polar-spherical coordinates with $r = |x_1 - y|$, to the fact that σ is positive homogeneous of order zero, to (4.1), and to definition (2.2).

A similar, simplified, argument shows that equation (4.6) holds by replacing x_1 by x_0 and 3δ by 2δ . So,

$$|I_1| \leq 2 \|\sigma\| [\phi]_{\overline{\omega}} \int_0^{3\delta} \frac{\overline{\omega}(r)}{r} dr \leq c \|\sigma\| [\phi]_{\overline{\omega}} \int_0^{\delta} \frac{\overline{\omega}(r)}{r} dr$$

where we have appealed to (3.17) for $k_2 = 3$. Hence,

$$(4.7) \quad |I_1| \leq c \|\sigma\| [\phi]_{\overline{\omega}} \widehat{\omega}(\delta).$$

On the other hand

$$\begin{aligned}
I_2 &= \int_{2\delta < |y-x_0| < \delta_0} (\phi(x_1) - \phi(x_0)) \mathcal{K}(x_0 - y) dy + \\
& \int_{2\delta < |y-x_0| < \delta_0} (\phi(y) - \phi(x_1)) (\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)) dy.
\end{aligned}$$

The first integral vanishes, due to (4.2). Hence,

$$|I_2| \leq \int_{2\delta < |y-x_0| < \delta_0} |\phi(y) - \phi(x_1)| |\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)| dy.$$

Further, by the mean-value theorem, there is a point x_2 , between x_0 and x_1 , such that

$$|\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)| \leq |\nabla \mathcal{K}(x_2 - y)| \delta.$$

Since

$$\partial_i \mathcal{K}(x) = \frac{1}{|x|^{n+1}} \left[(\partial_i \sigma) \left(\frac{x}{|x|} \right) - n \frac{x_i}{|x|} \sigma(x) \right],$$

it readily follows that

$$\begin{aligned}
& |\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)| \leq \\
(4.8) \quad & c \|\sigma\| \frac{\delta}{|y-x_2|^{n+1}} \leq c \|\sigma\| \frac{\delta}{|y-x_0|^{n+1}},
\end{aligned}$$

where $\|\sigma\|$ denotes the sum of the L^∞ norms of σ and of its first order derivatives on the surface of the unit sphere $I(0, 1)$. Note that, for $|x_0 - y| > 2\delta$, one has

$$|x_0 - y| \leq 2|x_2 - y| \leq 4|x_0 - y|.$$

On the other hand, for $2\delta < |x_0 - y|$,

$$|x_1 - y| \leq 3|x_0 - y|.$$

So,

$$|\phi(y) - \phi(x_1)| \leq [\phi]_{\overline{\omega}} \overline{\omega}(3|x_0 - y|).$$

The above estimates show that

$$\begin{aligned} |I_2| &\leq c \|\sigma\| [\phi]_{\overline{\omega}} \delta \int_{2\delta}^{\delta_0} \overline{\omega}(3r) r^{-2} dr \\ (4.9) \quad &\leq c \|\sigma\| [\phi]_{\overline{\omega}} \delta \int_{2\delta}^{\delta_0} \overline{\omega}(r) r^{-2} dr, \end{aligned}$$

where we appealed to (3.17) for $k_2 = 3$. Finally, by (3.10), it readily follows that

$$(4.10) \quad |I_2| \leq c \|\sigma\| [\phi]_{\overline{\omega}} \widehat{\omega}(\delta)$$

for $\delta \in (0, \delta_0)$.

Finally we consider I_3 . By arguing as for I_2 , in particular by appealing to (4.2) and (4.8), one shows that

$$\begin{aligned} (4.11) \quad |I_3| &\leq C \delta \|\sigma\| \int_{|y-x_0| > \delta_0} \frac{|\phi(y) - \phi(x_1)|}{|y-x_0|^{n+1}} dy \leq \\ &C \delta \|\sigma\| \|\phi\| \leq C \|\sigma\| \|\phi\| \widehat{\omega}(\delta). \end{aligned}$$

Note that, by a de l'Hôpital rule, one shows that (3.5) holds with $\overline{\omega}(r)$ replaced by $\widehat{\omega}(r)$. From equation (4.5), by appealing to (4.7), (4.10), and (4.11), one shows that

$$(4.12) \quad |(\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1)| \leq C \|\sigma\| \|\phi\|_{\overline{\omega}} \widehat{\omega}(\delta),$$

for each couple of points $x_0, x_1 \in I(\rho)$ such that $0 < |x_0 - x_1| \leq \delta_0$. Hence (4.1) holds.

We may easily estimate $|(\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1)|$ for pairs of points x_0, x_1 for which $\delta_0 < |x_0 - x_1| < \rho$. However this is superfluous, since δ_0 is fixed "once and for all". \square

5 The interior regularity estimate in the constant coefficients case.

In this chapter we apply the theorem 4.1 to prove the basic interior regularity result for solutions of the elliptic equation (1.1) in the framework of $D_{\overline{\omega}}$ data spaces. In this section \mathbf{L} is a constant coefficients operator. The proof is inspired by that developed in Hölder spaces in [7], part II, section 5. For convenience, assume that $n \geq 3$.

By a fundamental solution of the differential operator \mathbf{L} one means a distribution $J(x)$ in \mathbb{R}^n such that

$$(5.1) \quad \mathbf{L} J(x) = \delta(x).$$

The celebrated Malgrange-Ehrenpreis theorem states that every non-zero linear differential operator with constant coefficients has a fundamental solution (see, for instance, [16], Chap. VI, sec. 10). We recall that the analogue for differential

operators whose coefficients are polynomials (rather than constants) is false, as shown by a famous Hans Lewy's counter-example.

In particular, for a second order elliptic operator with constant coefficients and only higher order terms, one can construct explicitly a fundamental solution $J(x)$ which satisfies the properties (i), (ii), and (iii), claimed in [7], namely,

- (i) $J(x)$ is a real analytic function for $|x| \neq 0$.
- (ii) For $n \geq 3$

$$(5.2) \quad J(x) = \frac{\sigma(x)}{|x|^{n-2}},$$

where $\sigma(x)$ is positive homogeneous of degree 0.

(iii) Equation (5.1) holds. In particular, for every sufficiently regular, compact supported, function v , one has

$$v(x) = \int J(x-y) (\mathbf{L}v)(y) dy.$$

For a second order elliptic operator as above, one has

$$(5.3) \quad J(x) = c \left(\sum A_{ij} x_i x_j \right)^{\frac{2-n}{2}},$$

where A_{ij} denotes the cofactor of a_{ij} in the determinant $|a_{ij}|$.

Following [7], we denote by \mathbf{S} the operator

$$(5.4) \quad (\mathbf{S}\phi)(x) = \int J(x-y) \phi(y) dy = (J * \phi)(x).$$

Note that, in the constant coefficients case, the operator \mathbf{T} introduced in reference [7] vanishes.

Point (iii) above (see also [7] "Lemma" A) shows that if v is compact supported and sufficiently regular (for instance of class C^2), then

$$(5.5) \quad v = \mathbf{S}\mathbf{L}v.$$

Due to the structure of the function $\sigma(x)$ appearing in equation (5.2), it readily follows that second order derivatives of $(\mathbf{S}\phi)(x)$ have the form $\partial_i \partial_j \mathbf{S}\phi = \mathcal{K}_{ij} * \phi$, where the \mathcal{K}_{ij} enjoy the properties described for singular kernels \mathcal{K} in section 4.

We write, in abbreviated form,

$$(5.6) \quad \nabla^2 \mathbf{S}\phi(x) = \int \mathcal{K}(x-y) \phi(y) dy,$$

where $\mathcal{K}(x)$ enjoys the properties described at the beginning of section 4. From (5.6) it follows that

$$\nabla^2 \mathbf{S}\mathbf{L}v = \int \mathcal{K}(x-y) \mathbf{L}v(y) dy.$$

Hence, by Theorem 4.1, one gets

$$(5.7) \quad [\nabla^2 \mathbf{S}\mathbf{L}v]_{\widehat{\omega}; 2\rho} \leq C [\mathbf{L}v]_{\overline{\omega}; 2\rho}.$$

By appealing to (5.5) we get the following result.

Proposition 5.1. *Assume that the differential operator \mathbf{L} has constant coefficients and that the oscillation function $\bar{\omega}$ satisfies assumptions (3.1), (3.5), (3.16), and (3.10). Let v be a support compact function $\in C^2(2\rho)$, such that $\mathbf{L}v \in D_{\bar{\omega}}(2\rho)$. Then*

$$(5.8) \quad [\nabla^2 v]_{\bar{\omega}; 2\rho} \leq C [\mathbf{L}v]_{\bar{\omega}; 2\rho}.$$

One has the following interior regularity result. For brevity we have consider two spheres of radius ρ and R , $R > \rho$, in the particular case $R = 2\rho$.

Theorem 5.1. *Assume that the hypothesis of proposition 5.1 hold. Further, let $u \in C^2(2\rho)$ be such that $\mathbf{L}u \in D_{\bar{\omega}}(2\rho)$. Then $\nabla^2 u \in D_{\bar{\omega}}(\rho)$, moreover*

$$(5.9) \quad [\nabla^2 u]_{\bar{\omega}; \rho} \leq C [\mathbf{L}u]_{\bar{\omega}; 2\rho} + c(\theta) \left(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho} \right) \frac{|x-y|}{\bar{\omega}(|x-y|)},$$

for some positive constant C , independent of ρ . In particular,

$$(5.10) \quad [\nabla^2 u]_{\bar{\omega}; \rho} \leq C [\mathbf{L}u]_{\bar{\omega}; 2\rho} + \frac{c(\theta)}{\rho^3} \|u\|_{C^2(2\rho)}.$$

Proof. Fix a no-negative C^∞ function θ , defined for $0 \leq t \leq 1$ such that $\theta(t) = 1$ for $0 \leq t \leq \frac{1}{3}$, and $\theta(t) = 0$ for $\frac{2}{3} \leq t \leq 1$. Further fix a positive real ρ , for convenience $0 < \rho < \frac{1}{2}$, and define

$$(5.11) \quad \zeta(x) = \begin{cases} 1 & \text{for } |x| \leq \rho, \\ \theta\left(\frac{|x|-\rho}{\rho}\right) & \text{for } \rho \leq |x| \leq 2\rho. \end{cases}$$

Next we consider $\zeta(x)$ for points x such that $\rho \leq |x| \leq 2\rho$, and leave to the reader different situations. Due to symmetry, it is sufficient to consider the one dimensional case

$$\zeta(t) = \theta\left(\frac{t-\rho}{\rho}\right) \quad \text{for } \rho \leq t \leq 2\rho.$$

Hence

$$\zeta'(t) = \theta'\left(\frac{t-\rho}{\rho}\right) \frac{1}{\rho},$$

and

$$\zeta''(t) = \theta''\left(\frac{t-\rho}{\rho}\right) \frac{1}{\rho^2}.$$

Further,

$$\rho^2 |\zeta''(t_2) - \zeta''(t_1)| \leq \left| \theta''\left(\frac{t_2-\rho}{\rho}\right) - \theta''\left(\frac{t_1-\rho}{\rho}\right) \right|,$$

where

$$\left| \frac{t_2-\rho}{\rho} - \frac{t_1-\rho}{\rho} \right| = \left| \frac{t_2-t_1}{\rho} \right| \leq \frac{1}{3} < 1.$$

So

$$(5.12) \quad |\zeta''(t_2) - \zeta''(t_1)| \leq \frac{1}{\rho^3} [\theta'']_{Lip} |t_2 - t_1|,$$

where $[\cdot]_{Lip}$ denotes the usual Lipschitz semi-norm.

Set

$$(5.13) \quad v = \zeta u.$$

Note that $\mathbf{L}v \in D_{\overline{\omega}}(2\rho)$, moreover the support of v is contained in $|x| < 2\rho$.

On the other hand,

$$(5.14) \quad \mathbf{L}v = \zeta \mathbf{L}u + N.$$

One has

$$\begin{aligned} |(\zeta \mathbf{L}u)(x) - (\zeta \mathbf{L}u)(y)| &\leq \|\zeta\| [\mathbf{L}u]_{\overline{\omega}} \overline{\omega}(|x - y|) + \|\nabla \zeta\| \|\mathbf{L}u\| |x - y| \\ &\leq [\mathbf{L}u]_{\overline{\omega}} \overline{\omega}(|x - y|) + c \|\theta'\| \frac{1}{\rho} \|\nabla^2 u\| |x - y|. \end{aligned}$$

Hence,

$$(5.15) \quad [\zeta \mathbf{L}u]_{\overline{\omega}} \leq [\mathbf{L}u]_{\overline{\omega}} + c \|\theta'\| \frac{1}{\rho} \|\nabla^2 u\| \frac{|x - y|}{\overline{\omega}(|x - y|)}.$$

Next we prove that

$$(5.16) \quad [N]_{\overline{\omega}} \leq c(\theta) \left(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho} \right) \frac{|x - y|}{\overline{\omega}(|x - y|)}.$$

One has

$$N \cong (\nabla^2 \zeta) u + (\nabla \zeta) (\nabla u) \equiv A + B.$$

By appealing in particular to (5.12), straightforward calculations show that

$$\begin{aligned} |A(x) - A(y)| &\leq \|\nabla u\| \|\nabla^2 \zeta\| |x - y| + \|u\| \frac{1}{\rho^3} [\theta'']_{Lip} |x - y| \\ &\leq \left(\frac{1}{\rho^2} \|\theta''\| \|\nabla u\| + \frac{1}{\rho^3} [\theta'']_{Lip} \|u\| \right) |x - y|. \end{aligned}$$

Hence

$$(5.17) \quad [A]_{\overline{\omega}} \leq c(\theta) \left(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} \right) \frac{|x - y|}{\overline{\omega}(|x - y|)}.$$

Similar manipulations show that

$$(5.18) \quad [B]_{\overline{\omega}} \leq c(\theta) \left(\frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho} \right) \frac{|x - y|}{\overline{\omega}(|x - y|)}.$$

Equation (5.16) follows from (5.17) and (5.18).

Lastly, from (5.14), (5.15), and (5.16) one shows that

$$(5.19) \quad [\mathbf{L}v]_{\overline{\omega}} \leq [\mathbf{L}u]_{\overline{\omega}} + c(\theta) \left(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho} \right) \frac{|x - y|}{\overline{\omega}(|x - y|)}.$$

In the following not labeled norms concern the domain $I(2\rho)$.

From (5.13), (5.5), (5.7), and (5.19) one gets

$$\begin{aligned} (5.20) \quad &[\nabla^2 u]_{\widehat{\omega}; \rho} \leq [\nabla^2 v]_{\widehat{\omega}} \leq C [\mathbf{L}v]_{\overline{\omega}} \\ &\leq C [\mathbf{L}u]_{\overline{\omega}} + c(\theta) \left(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho} \right) \frac{|x - y|}{\overline{\omega}(|x - y|)}, \end{aligned}$$

where $0 < 2\rho < 1$. □

6 The interior regularity estimate in the variable coefficients case.

In this section we extend the estimate (5.9) to uniformly elliptic operators with variable coefficients

$$(6.1) \quad \mathbf{L} = \sum_1^n a_{ij}(x) \partial_i \partial_j.$$

To avoid non significant manipulations we assume that the coefficients $a_{ij}(x)$ are Lipschitz continuous in $I(2\rho)$, which Lipschitz constants bounded by a constant A . Following the same belief, we left to the reader the introduction of lower order terms.

We assume that

$$(6.2) \quad \bar{\omega}(r) \leq k_1 \hat{\omega}(r),$$

for some positive constant k_1 , and r in some neighborhood of the origin. This yields $D_{\bar{\omega}}(\bar{\Omega}) = D_{\hat{\omega}}(\bar{\Omega})$, recall proposition 2.1. Assumption (6.2) holds if in equation (3.7) the constant C_1 is finite. In fact,

$$\lim_{r \rightarrow 0} \frac{\bar{\omega}(r)}{\hat{\omega}(r)} = \lim_{r \rightarrow 0} \frac{r \bar{\omega}'(r)}{\bar{\omega}(r)} = \frac{1}{C_1},$$

if the second limit exists.

In the following we appeal to the constant coefficients operator

$$(6.3) \quad \mathbf{L}_0 = \sum_1^n b_{ij} \partial_i \partial_j,$$

where $b_{ij} = a_{ij}(0)$. Clearly,

$$(6.4) \quad \mathbf{L}_0 v(x) = \mathbf{L} v(x) + (\mathbf{L}_0 - \mathbf{L}) v(x).$$

One has

$$(6.5) \quad \begin{aligned} & (\mathbf{L}_0 - \mathbf{L}) v(x) - (\mathbf{L}_0 - \mathbf{L}) v(y) = \\ & ((b_{ij} - a_{ij}(x)) (\partial_{ij}^2 v(x) - \partial_{ij}^2 v(y)) + ((a_{ij}(y) - a_{ij}(x)) (\partial_{ij}^2 v(y) \end{aligned}$$

where, for convenience, summation on repeated indexes is assumed. Straight-forward calculations easily lead to the following pointwise estimate

$$(6.6) \quad |(\mathbf{L}_0 - \mathbf{L}) v(x) - (\mathbf{L}_0 - \mathbf{L}) v(y)| \leq c A (2\rho [\nabla^2 v]_{\bar{\omega}} + \|\nabla^2 v\| \frac{|x-y|}{\bar{\omega}(|x-y|)}) \bar{\omega}(|x-y|),$$

where norms and semi-norms concern the sphere $I(0, 2\rho)$.

Next assume that $v \in C^2(2\rho)$ has compact support in $I(0, 2\rho)$, and $\mathbf{L} v \in D_{\bar{\omega}}(2\rho)$. Then, by (6.4), (6.5), and (5.8) it follows that

$$[\nabla^2 v]_{\bar{\omega}; 2\rho} \leq C [\mathbf{L} v]_{\bar{\omega}; 2\rho} + C \rho [\nabla^2 v]_{\bar{\omega}; 2\rho} + C \|\nabla^2 v\| \frac{|x-y|}{\bar{\omega}(|x-y|)}.$$

In particular

$$(6.7) \quad [\nabla^2 v]_{\bar{\omega}; 2\rho} \leq C [\mathbf{L} v]_{\bar{\omega}; 2\rho} + C \rho [\nabla^2 v]_{\bar{\omega}; 2\rho} + C \|\nabla^2 v\|.$$

Now, from (6.2), one gets

$$(6.8) \quad (1 - C k_1 \rho) [\nabla^2 v]_{\bar{\omega}; 2\rho} \leq C ([\mathbf{L} v]_{\bar{\omega}; 2\rho} + \|\nabla^2 v\|).$$

Next we set

$$v = \zeta u$$

and argue as done to prove (5.9). This proves the following result, in the case of variable coefficients operators.

Theorem 6.1. *Assume that the oscillation function $\bar{\omega}$ satisfies conditions (3.1), (3.5), (3.16), (3.10), and (6.2). Further, assume that*

$$0 < \rho \leq \frac{1}{2Ck_1},$$

and let $\mathbf{L} u \in D_{\bar{\omega}}(2\rho)$, for some $u \in C^2(2\rho)$. Then $\nabla^2 u \in D_{\bar{\omega}}(\rho)$, and

$$(6.9) \quad [\nabla^2 u]_{\bar{\omega}; \rho} \leq C [\mathbf{L} u]_{\bar{\omega}; 2\rho} + \frac{C}{\rho^3} \|u\|_{C^2(2\rho)},$$

for suitable positive constants C , independent of ρ .

7 Proof of theorems 3.2 and 3.3.

The local estimates (estimates in Ω_0 , $\Omega_0 \subset \subset \Omega$) claimed in theorems 3.2 and 3.3 follow immediately from the interior estimates, by appealing to the classical method consisting in covering $\bar{\Omega}_0$ by a finite number of sufficiently small spheres. For brevity, we may estimate the quantities originated by the terms $\|u\|_{C^2(2\rho)}$, see the right hand sides of equations (5.10) and (6.9), simply by appealing to the theorem 1.1, which shows that solutions u satisfy the estimate

$$(7.1) \quad \|u\|_{C^2(\bar{\Omega})} \leq c \|f\|_*.$$

Concerning regularity up to the boundary one proceeds as follows. The main point, the extension of the interior regularity estimate (5.10) from spheres to half-spheres, is obtained by following the argument described in part II, section 5.6, reference [7]. One starts by showing that the interior estimate in spheres also hold for half-spheres, under the zero boundary condition on the flat part of the boundary. One appeals to "reflection" of u through the flat boundary, as an odd function, in the orthogonal direction, from the half to the whole sphere. In this way the half-sphere problem goes back to an whole-sphere problem, absolutely similar to that considered in section 5, see [7]. Note that is sufficient, and more convenient, to show this extension to half-spheres merely for constant coefficient operators. The regularity result "up to flat boundary points" claimed in theorem 3.2 follows.

Extension of the half-sphere's estimate, from constant coefficients to variable coefficients, is obtained exactly as done in section 6 for whole spheres. Obviously, this requires assumption (6.2). Then, sufficiently small neighborhoods

of boundary points are regularly mapped, one to one, onto half-spheres. This procedure allows extension of the local estimate for functions u defined on sufficiently small neighborhoods of boundary points, vanishing on the boundary. A well known finite covering argument leads to the thesis of theorem 3.3.

The above extension to non-flat boundary points requires local changes of coordinates. This transforms constant coefficients in variable coefficients operators. So, local regularity up to non-flat boundary points for constant coefficients operators can not be claimed here. This is a challenging open problem. One may start by considering the particular case of data in Log spaces.

We note that in the proof of Theorem 1, section 5.4, part II, in reference [7], density of C^1 in Hölder spaces is used. The same occurs in the proof of lemma B, section 5.3.

8 The Log spaces $D^{0,\alpha}(\overline{\Omega})$. An intermediate regularity result.

The following is a significant example of functional space $D_{\overline{\omega}}(\overline{\Omega})$ which yields intermediate (not full) regularity, based on the well known formulae

$$(8.1) \quad \int \frac{(-\log r)^{-\alpha}}{r} dr = \frac{1}{\alpha-1} (-\log r)^{1-\alpha},$$

where $0 < \alpha < +\infty$ (for $\alpha = 1$ the right hand side should be replaced by $-\log(-\log r)$). Equation (8.1) shows that the $C_*(\overline{\Omega})$ semi-norm (1.5) is finite if

$$(8.2) \quad \omega_f(r) \leq C (-\log r)^{-\alpha},$$

for some $\alpha > 1$ and some constant $C > 0$. This led to define, for each fixed $\alpha > 0$, the semi-norm

$$(8.3) \quad [f]_{\alpha} \equiv \sup_{r \in (0,1)} \frac{\omega_f(r)}{\omega_{\alpha}(r)},$$

where the *oscillation function* $\omega_{\alpha}(r)$ is defined by setting

$$(8.4) \quad \omega_{\alpha}(r) = (-\log r)^{-\alpha}.$$

Hence $[f]_{\alpha}$ is the smallest constant for which the estimate

$$(8.5) \quad |f(x) - f(y)| \leq [f]_{\alpha} \cdot \left(\log \frac{1}{|x-y|} \right)^{-\alpha}$$

holds for all couple $x, y \in \overline{\Omega}$ such that $|x-y| < 1$. Note that we have merely replaced, in the definition of Hölder spaces, the quantity

$$\frac{1}{|x-y|} \quad \text{by} \quad \log \frac{1}{|x-y|},$$

and allow α to be arbitrarily large.

Definition 8.1. For each real positive α , we set

$$(8.6) \quad D^{0,\alpha}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_\alpha < \infty\}.$$

A norm is introduced in $D^{0,\alpha}(\overline{\Omega})$ by setting $\|f\| \equiv [f]_\alpha + \|f\|$.

We call these spaces Log spaces. We remark that in reference [5] we have called these spaces H-log spaces.

The restriction $|x - y| < 1$ in equation (8.3) is due to the behavior of the function $\log r$, for $r \geq 1$. Note that, by replacing $0 < |x - y| < 1$ by $0 < |x - y| < \rho$ in equation (8.3), for some $0 < \rho < 1$, it follows that

$$(8.7) \quad [f]_{\alpha;\rho} \leq [f]_\alpha \leq [f]_{\alpha;\rho} + \frac{2}{(-\log \rho)^{-\alpha}} \|f\|,$$

where the meaning of $[f]_{\alpha;\rho}$ seems clear. Hence, the norms $\|f\|_\alpha$ and $\|f\|_{\alpha;\rho}$ are equivalent. We may also avoid the above $|x - y| < 1$ inconvenient by replacing in the denominator of the right hand side of (8.3) the quantity r by r/R , where $R = \text{diam } \Omega$, and by letting $r \in (0, R)$. We rather prefer the first definition, since the second one implies more ponderous notation.

For $0 < \beta < \alpha$, and $0 < \lambda \leq 1$, the (compact) embedding

$$(8.8) \quad C^{0,\lambda}(\overline{\Omega}) \subset D^{0,\alpha}(\overline{\Omega}) \subset D^{0,\beta}(\overline{\Omega}) \subset C(\overline{\Omega})$$

hold. Furthermore, for $1 < \alpha$, one has the (compact) embedding $D^{0,\alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega})$. Note that $D^{0,1}(\overline{\Omega}) \not\subset C_*(\overline{\Omega})$.

The properties proved in reference [5] for $D^{0,\alpha}(\overline{\Omega})$ spaces follow here from that proved for $D_{\overline{\omega}}(\overline{\Omega})$ spaces, since $\omega_\alpha(r)$ is a particular case of function $\overline{\omega}(r)$. It is worth noting that in reference [5] we claimed, and left the proof to the reader, that $C^\infty(\overline{\Omega})$ is dense in $D^{0,\alpha}(\overline{\Omega})$. Actually, as shown in theorem 2.5, this result is false.

The following result is a particular case of theorem 3.2.

Theorem 8.1. Let $\Omega_0 \subset\subset \Omega$, $f \in D^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 1$, and u be the solution of problem (1.1), where \mathbf{L} has constant coefficients. Then $\nabla^2 u \in D^{0,\alpha-1}(\Omega_0)$, moreover

$$(8.9) \quad \|\nabla^2 u\|_{\alpha-1,\Omega_0} \leq C \|f\|_\alpha,$$

for some positive constant $C = C(\alpha, \Omega_0, \Omega)$. The regularity result holds up to flat boundary points. Results are optimal in the sharp sense, see section 10. In particular, for $\beta > \alpha - 1$, $\nabla^2 u \in D^{0,\beta}(\Omega_0)$ is false in general.

Theorem 8.1 is a particular case of theorem 3.2. In fact, the oscillation function $\omega_\alpha(r)$ is concave and differentiable for $r > 0$, satisfies (3.1) for $\alpha > 1$, and (3.5) holds. Further, condition (3.7) follows from

$$(8.10) \quad \lim_{r \rightarrow 0} \frac{\omega_\alpha(r)}{r \omega'_\alpha(r)} = +\infty.$$

In reference [5] the above regularity result was claimed up to the boundary. However the proof is not complete, since extension to non-flat boundary points requires estimates for variable coefficients operators. The reason for this requirement was explained in section 7.

Next we apply to the results stated in theorem 8.1 to illustrate, by means of a simple example, the meaning of *sharp* optimality. This concept will be discussed in a more abstract form in section 10. Optimality of regularity results is not confined here to the particular family of spaces under consideration, but is something stronger. Let us illustrate the distinction. The theorem 8.1 claims that $\nabla^2 u \in D^{0, \alpha-1}(\overline{\Omega})$. Optimality *restricted* to the Log spaces framework means that, given $\beta > \alpha - 1$, there is at least a data $f \in D^{0, \alpha}(\overline{\Omega})$ for which $\nabla^2 u$ does not belong to $D^{0, \beta}(\overline{\Omega})$. This situation does not exclude that (for instance, and to fix ideas) for all $f \in D^{0, \alpha}(\overline{\Omega})$ the oscillation $\omega(r)$ of $\nabla^2 u$ satisfies the stronger estimate

$$(8.11) \quad \omega(r) \leq C_f \left[\log \left(\log \frac{1}{r} \right) \right]^{-1} \cdot (-\log r)^{-(\alpha-1)}.$$

In fact, for each $\beta > \alpha - 1$, one has

$$(-\log r)^{-\beta} << \left[\log \left(\log \frac{1}{r} \right) \right]^{-1} \cdot (-\log r)^{-(\alpha-1)} << (-\log r)^{-(\alpha-1)}.$$

Sharp optimality avoids the above, and similar, possibilities. This fact is significant in all cases in which full regularity is not reached, as in Theorem 8.1. This is the meaning giving here to the sharpness of a regularity result.

Concerning references, not related to our results but merely to Log spaces (mostly for $n = 1$, or $\alpha = 1$), the author is grateful to Francesca Crispo for calling our attention to the treatise [9], to which the reader is referred. In particular, as claimed in the introduction of this volume, the space $D^{0, 1}(\overline{\Omega})$ was considered in reference [14]. See also definition 2.2 in reference [9]. Other references, quoted in [9], are [10], [12], [17], [18], and [19].

9 Hölog spaces $C_{\alpha}^{0, \lambda}(\overline{\Omega})$ and full regularity.

If, for some $\lambda > 0$, one has $\widehat{\omega}(r) = \lambda \overline{\omega}(r)$ in a neighborhood of the origin, then there is a $k > 0$ such that $\overline{\omega}(r) = k r^{\lambda}$. This fact could suggest that Hölder spaces could be the unique full regularity class inside our framework. However, *full regularity* is also enjoyed by other spaces. The following is a particularly interesting case. Consider oscillation functions

$$(9.1) \quad \omega_{\lambda, \alpha}(r) = r^{\lambda} (-\log r)^{-\alpha}, \quad r < 1,$$

where $0 \leq \lambda < 1$ and $\alpha \in \mathbb{R}$. For $\lambda = 0$ and $\alpha > 0$ we re-obtain the Log space $D^{0, \alpha}(\overline{\Omega})$, for $\lambda > 0$ and $\alpha = 0$ we re-obtain $C^{0, \lambda}(\overline{\Omega})$. Theorem 2.3 shows that (compact) inclusions

$$C^{0, \lambda_2}(\overline{\Omega}) \subset C_{\alpha}^{0, \lambda}(\overline{\Omega}) \subset C_{\beta}^{0, \lambda}(\overline{\Omega}) \subset C^{0, \lambda}(\overline{\Omega}) \subset C_{-\beta}^{0, \lambda}(\overline{\Omega}) \subset C_{-\alpha}^{0, \lambda}(\overline{\Omega}) \subset C^{0, \lambda_1}(\overline{\Omega})$$

hold for $\alpha > \beta > 0$ and $0 < \lambda_1 < \lambda < \lambda_2 < 1$. The reader should note that the set

$$\bigcup_{\lambda, \alpha} C_{\alpha}^{0, \lambda}(\overline{\Omega}),$$

where $0 < \lambda < 1$, and $\alpha \in \mathbb{R}$, is a *totally* ordered set, in the obvious way. In the totally ordered sub-chain merely consisting of classical Hölder spaces, each $C^{0, \lambda}$

space can be enlarged, to become an infinite, ordered chain, $C_{\alpha}^{0,\lambda}(\overline{\Omega})$, $\alpha \in \mathbb{R}$. Clearly, the spaces $C_{\alpha}^{0,\lambda}(\overline{\Omega})$ enjoy all the interesting properties described in section 2.

To abbreviate notation, in this section we set

$$\overline{\omega}(r) \equiv \omega_{\lambda,\alpha}(r), \quad [f]_{\overline{\omega}} \equiv [f]_{\lambda,\alpha}, \quad \text{and} \quad \|f\|_{\overline{\omega}} \equiv \|f\|_{\lambda,\alpha}.$$

The following full regularity result holds.

Theorem 9.1. *Let $f \in C_{\alpha}^{0,\lambda}(\overline{\Omega})$ for some $\lambda \in (0, 1)$ and some $\alpha \in \mathbb{R}$. Let u be the solution of problem (1.1), where the differential operator \mathbf{L} may have variable coefficients. Then $\nabla^2 u \in C_{\alpha}^{0,\lambda}(\overline{\Omega})$. Moreover*

$$(9.2) \quad \|\nabla^2 u\|_{\lambda,\alpha} \leq C \|f\|_{\lambda,\alpha},$$

for some positive constant C . The result is optimal, in the sharp sense.

Note that full regularity $\omega_{\lambda,\alpha} \rightarrow \omega_{\lambda,\alpha}$ is a little surprising here. In fact, at the light of theorem 8.1, we could merely expected the intermediate regularity result $\omega_{\lambda,\alpha} \rightarrow \omega_{\lambda,\alpha-1}$.

Proof. We appeal to the theorem 3.2. Assumptions (3.1) and (3.5) are trivially verified. Let's prove (3.7). Set

$$L(r) = \log \frac{1}{r}.$$

Straightforward calculations show that

$$(9.3) \quad \overline{\omega}'(r) = r^{\lambda-1} L(r)^{-\alpha} (\lambda + \alpha L(r)^{-1})$$

and that

$$(9.4) \quad \overline{\omega}''(r) = -r^{\lambda-2} L(r)^{-\alpha} \left(\lambda(1-\lambda) - (2\lambda-1)\alpha L(r)^{-1} - \alpha(\alpha+1)L(r)^{-2} \right).$$

Equation (9.4) shows that $\overline{\omega}''(r) < 0$ in a neighborhood of the origin, since $\lim_{r \rightarrow 0} L(r) = +\infty$. Hence $\overline{\omega}$ is concave. Furthermore (3.7) holds since

$$(9.5) \quad \lim_{r \rightarrow 0} \frac{\overline{\omega}(r)}{r \overline{\omega}'(r)} = \frac{1}{\lambda} > 1.$$

To prove full regularity we appeal to de l'Hôpital rule and to (9.5) to show that

$$(9.6) \quad \lim_{r \rightarrow 0} \frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r \rightarrow 0} \frac{\overline{\omega}(r)}{r \overline{\omega}'(r)} = \frac{1}{\lambda}.$$

In particular (2.5) holds for r in some neighborhood of the origin. Hence proposition 2.1 applies. \square

It would be interesting to study higher order regularity results in the framework of Hölog spaces.

10 Sharpness of the regularity results.

In this section we prove the *sharpness* of our regularity results (a simple example was shown at the end of section 8). The proof is quite adaptable to different situations, local and global results, etc. We merely show the main argument. We construct a counter-example, which concerns constant coefficients operators (we could easily deny case by case), which shows that any stronger regularity result can not occur. We start by considering the Laplace operator Δ . We remark that the argument applies to the regularity results stated in theorems 3.2 and 3.3. However, in the second theorem, the conclusion is obvious, due to full regularity.

For convenience, we assume that $\bar{\omega}(r)$ is differentiable, and that there is a positive constant C such that

$$(10.1) \quad \frac{\bar{\omega}(r)}{r \bar{\omega}'(r)} \geq C > 0,$$

for $r > 0$, in a neighborhood of the origin. Note that (10.1) holds, with $C = 1$, if $\bar{\omega}(r)$ is concave.

Proposition 10.1. *Assume that $\bar{\omega}(r)$ satisfies the above hypothesis, and let $\hat{\omega}_0(r)$ be a given oscillation function. Assume that the results stated in theorem 3.2 hold by replacing $\hat{\omega}$ by $\hat{\omega}_0$. Then there is a constant c for which $\hat{\omega}(r) \leq c \hat{\omega}_0(r)$.*

We may say that any regularity result better than (8.9) is false.

Proof. For simplicity, we start by assuming that $\mathbf{L} = \Delta$. Consider the function

$$(10.2) \quad u(x) = \hat{\omega}(|x|) x_1 x_2,$$

defined in \mathbb{R}^n , $n \geq 2$. Actually, we are merely interested in the behavior near the origin (see (10.6) below).

Straightforward calculations show that

$$(10.3) \quad \Delta u(x) = (n+2) \frac{x_1 x_2}{|x|^2} \bar{\omega}(x) + \frac{x_1 x_2}{|x|^2} |x| \bar{\omega}'(|x|).$$

In particular, $\Delta u(0) = 0$. By appealing to (10.1) one shows that

$$|\Delta u(x) - \Delta u(0)| = |\Delta u(x)| \leq C \bar{\omega}(|x|).$$

Hence, in a neighborhood of the origin, $f(x) = \Delta u(x)$ belongs to $D_{\bar{\omega}}$.

On the other hand, straightforward calculations show that

$$(10.4) \quad \partial_1 \partial_2 u(x) = \hat{\omega}(|x|) + \frac{1}{|x|^2} (x_1^2 + x_2^2 - 2 \frac{x_1^2 x_2^2}{|x|^2}) \cdot \bar{\omega}(|x|) + \frac{x_1^2 x_2^2}{|x|^4} \cdot (|x| \bar{\omega}'(|x|)).$$

In particular $\partial_1 \partial_2 u(0) = 0$, and

$$|\partial_1 \partial_2 u(x) - \partial_1 \partial_2 u(0)| \geq \hat{\omega}(|x|)$$

for $0 < |x| \ll 1$, since in equation (10.4) the coefficients of $\bar{\omega}(|x|)$ and of $|x| \bar{\omega}'(|x|)$ are nonnegative. On the other hand, if $\hat{\omega}_0(r)$ regularity holds, one has

$$|\partial_1 \partial_2 u(x) - \partial_1 \partial_2 u(0)| \leq (c \|f\|_{\bar{\omega}}) \hat{\omega}_0(|x|)$$

for some $c > 0$. Hence $\widehat{\omega}(r) \leq c_0 \widehat{\omega}_0(r)$, for $r > 0$, in a neighborhood of the origin.

If \mathbf{L} is given by (1.2) we replace (10.2) by

$$(10.5) \quad u(x) = \widehat{\omega}(|x|) \sum_{i,j=1}^n b_{ij} x_i x_j,$$

where $B \neq 0$ is symmetric and

$$\sum_{i,j=1}^n a_{ij} b_{ij} = 0.$$

In particular, if a specific coefficient a_{kl} vanishes, we may simply choose $u(x) = \widehat{\omega}(|x|) x_k x_l$, as done in (10.2).

We localize the above result as follows. Assume that $0 \in \Omega$, and consider the function

$$(10.6) \quad u(x) = \psi(|x|) \widehat{\omega}(|x|) x_1 x_2,$$

where $\psi(r)$ is non-negative, indefinitely differentiable, vanishes for $r \geq \rho > 0$, and is equal to 1 for $|x| < \frac{\rho}{2}$. The radius ρ is such that $I(0, \rho)$ is contained in Ω . The above truncation allows us to assume homogeneous boundary conditions in Ω (we may consider combinations of functions as above, centered in different points in Ω , with distinct radius, and distinct cut-off functions). \square

It is worth noting that in the above argument the specific expressions of the coefficients of $\overline{\omega}(|x|)$ and $|x| \overline{\omega}'(|x|)$ are secondary (even if the non-negativity of these coefficients was exploited). They are homogeneous functions of degree zero, without particular influence on the minimal regularity. The crucial point is that the second order derivative $\partial_1 \partial_2 u(x)$, due to the term $x_1 x_2$ in (10.2), leaves unchanged the "bad term" $\widehat{\omega}(|x|)$. This does not occur for derivatives $\partial_i^2 u(x)$, hence does not occur for $\Delta u(x)$.

It looks interesting to note that the "bad term" $\widehat{\omega}(|x|)$ can not be eliminated by the other two terms which are present in the right hand side of (10.4). Even when full regularity occurs (like in Hölder and Hölog spaces), the "bad term" $\widehat{\omega}(|x|)$ is still not eliminated. It simply is as regular as the other two terms, $\overline{\omega}(|x|)$ and $|x| \overline{\omega}'(|x|)$. See also [6], section 6, for some comment.

11 On data spaces larger than $C_*(\overline{\Omega})$.

In the context of [1], Theorem 1.1 was peripheral. Hence, the proof (written in a still existing manuscript, denoted here [BVUN]), remained unpublished. Actually, at that time, we proved the above result for more general elliptic boundary value problems. The proofs depend only on the behavior of the related Green's functions. Recently, by following the same ideas, we have shown, see [2], that for every $\mathbf{f} \in C_*(\overline{\Omega})$ the solution (\mathbf{u}, p) to the Stokes system

$$(11.1) \quad \begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma \end{cases}$$

belongs to $C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$.

In the manuscript [BVUN] we also tried to extend the result claimed in theorem 1.1 to data belonging to functional spaces $B_*(\overline{\Omega})$ containing $C_*(\overline{\Omega})$. By setting

$$\omega_f(x; r) = \sup_{y \in \Omega(x; r)} |f(x) - f(y)|,$$

we may write

$$(11.2) \quad [f]_* = \int_0^R \sup_{x \in \overline{\Omega}} \omega_f(x; r) \frac{dr}{r}.$$

So, together with $C_*(\overline{\Omega})$, we have considered a functional space $B_*(\overline{\Omega})$ obtained by commuting *integral* and *sup* operators in the right hand side of definition (11.2): For each $f \in C(\overline{\Omega})$, we defined the semi-norm

$$(11.3) \quad \langle f \rangle_* = \sup_{x \in \overline{\Omega}} \int_0^R \omega_f(x; r) \frac{dr}{r},$$

and a related functional space $B_*(\overline{\Omega})$. We have shown that the inclusion $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega})$ is proper, by constructing strongly oscillating functions which belong to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$. This construction was recently published in reference [3], Proposition 1.7.1. Furthermore, in [BVUN], we have shown that Theorem 1.1 and similar results hold in a weaker form, for data $f \in B_*(\overline{\Omega})$, by proving that the first order derivatives of the solution u are Lipschitz continuous in $\overline{\Omega}$. The proof is published in reference [3], actually for data in a functional space $D_*(\overline{\Omega})$ containing $B_*(\overline{\Omega})$. See Theorem 1.3.1 in [3]. A similar extension holds for the Stokes problem, as shown in reference [4], Theorem 6.1, where we have proved that if $\mathbf{f} \in \mathbf{D}_*(\overline{\Omega})$, then the solution (\mathbf{u}, p) of problem (11.1) satisfies the estimate $\|\mathbf{u}\|_{1,1} + \|p\|_{0,1} \leq C \|\mathbf{f}\|_*$. Full regularity for data in $\mathbf{B}_*(\overline{\Omega})$ would follow from a possible density of regular functions in this last space, a challenging open problem. The simple proof would be obtained by replacing the space $C_*(\overline{\Omega})$ by $B_*(\overline{\Omega})$ in [2], section 4. A similar remark holds for $\mathbf{D}_*(\overline{\Omega})$. However, in this last case, the desired density result looks quite unlikely.

References

- [1] H. Beirão da Veiga, *On the solutions in the large of the two-dimensional flow of a nonviscous incompressible fluid*, J. Diff. Eq., 54, (1984), no.3, 373-389.
- [2] H. Beirão da Veiga, *Concerning the existence of classical solutions to the Stokes system. On the minimal assumptions problem*, J. Math. Fluid Mech., 16 (2014), 539-550.
- [3] H. Beirão da Veiga, *Classical solutions to the two-dimensional Euler equations and elliptic boundary value problems, an overview*, in "Recent Progress in the Theory of the Euler and Navier-Stokes Equations", London Math. Soc. Lecture Notes, Edited by J. C. Robinson, J. L. Rodrigo, W. Sadowski, and A. V. López. Cambridge University Press, 2016.

- [4] H. Beirão da Veiga, *On some regularity results for the stationary Stokes system, and the 2 – D Euler equations*, Portugaliae Math., 72 (2015) 285-307.
- [5] H. Beirão da Veiga, *H-log spaces of continuous functions, potentials, and elliptic boundary value problems*, ArXiv 1503.04173 [math.AP], 13.03.2015.
- [6] H. Beirão da Veiga, *Elliptic boundary value problems in spaces of continuous functions, to appear in a special volume of DCDS Series S*.
- [7] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, John Wiley and Sons, Inc., New-York, 1964.
- [8] C.C. Burch, *The Dini condition and regularity of weak solutions of elliptic equations*, J. Diff. Eq., 30, (1978), 308-323.
- [9] D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces Foundations and Harmonic Analysis*, Springer, Basel 2013.
- [10] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$* , Math. Inequal. Appl., 7(2):245253, 2004.
- [11] O.A. Ladyzenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New-York, 1969.
- [12] S. Samko, *Convolution type operators in $L^{p(x)}$* , Integral Transform, Spec. Funct., 7(12):123 144, 1998.
- [13] V.L. Shapiro, *Generalized and classical solutions of the nonlinear stationary Navier-Stokes equations*, Trans. Amer. Math. Soc., 216 (1976) 61-79.
- [14] I. I. Sharapudinov, *The basis property of the Haar system in the space $L^{p(t)}[0, 1]$, and the principle of localization in the mean*, Mat. Sb. (N.S.), 130(172)(2):275283, 286, 1986.
- [15] V.A. Solonnikov, *On estimates of Green's tensors for certain boundary value problems*, Doklady Akad. Nauk. **130** (1960), 128-131.
- [16] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin-Heidelberg, 2nd edition, 1968.
- [17] V. V. Zhikov, *On the homogenization of nonlinear variational problems in perforated domains*, Russian J. Math. Phys., 2(3):393408, 1994.
- [18] V. V. Zhikov, *On Lavrentievs phenomenon*, Russian J. Math. Phys., 3(2):249269, 1995.
- [19] V. V. Zhikov, *On some variational problems*, Russian J. Math. Phys., 5(1):105116 (1998), 1997.